Neural coding and cognitive neuroscience in general concerns input-output relationships.

**Inputs**
- Light intensity
- Pre-synaptic action potentials
- Number of items in display

**Outputs**
- Number of isomerizations
- post-synaptic conductance
- Search time
Linear Systems

- To a first approximation, we can often model these relationships as linear systems.
- This makes it much easier to
  - Identify these systems (next topic)
  - Make predictions
    - e.g.,
      - given a new input $x$, the output will be $y$
      - given an observed output $y$, the input must have been $x$
Don’t get carried away

- Clearly the brain cannot be one big linear system:
  - We do not simply ‘resonate’ with our environment.
  - We have to make decisions: this involves the nonlinear mapping of observations to categorical variables.
  - However, many subsystems can be approximated as linear.
What is a linear system?

- Consider a system $T$ that maps inputs $x$ to outputs $y$.
- The system is linear if it satisfies the principle of superposition:

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$
Shift Invariance

- We will typically be concerned with inputs $x$ defined over time $t$ and/or space $u$.
- A system $T$ is **shift-invariant** with respect to one of these variables if a shift in the input along the variable produces an identical shift in the output:

$$y(t) = T[x(t)] \rightarrow y(t-s) = T[x(t-s)]$$

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**Diagram:**
- Original input
- Electrical Activity
- Output
- Original, later in time
- Electrical Activity
- Output, later in time
Continuous and Discrete Signals

- Most signals are discrete at a fine scale, and our measurements are always discrete.
- However, we can often approximate these as continuous, and this is sometimes convenient mathematically.
Pulses and Steps

Discrete pulse

\[ \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases} \]

Dirac delta function

\[ \delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \]

Unit step

\[ u(t) = \int_{-\infty}^{t} \delta(s) \, ds \]

\[ u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \]
Representing a signal with impulses

The two basic tests of linearity are homogeneity and additivity. To see whether a system is linear, we need to test whether it obeys certain rules that all linear systems obey. The trick is that original system we started with, but in fact, we will very shortly be able to use Eq. 1 to perform a marvelous approximation.

Digital compact disc, for example, stores whole complex pieces of music as lots of simple numbers representing very short impulses, and then the CD player adds all the impulses back together one after another to recreate the complex musical waveform.

In other words, we can represent any signal as an infinite sum of shifted and scaled unit impulses. A signal

\[ x(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_\Delta(t - k\Delta) \Delta \]

equals

\[ x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_\Delta(t - k\Delta) \Delta = \int_{-\infty}^{\infty} x(s)\delta(t - s) ds \]
The superposition principle is very powerful. It means that we can characterize a linear system \( T \) completely by its response to a unit impulse.

In particular, we can use the impulse response function to predict the response of the system to any input.
Convolution and the Impulse Response Function

\[ y(t) = T[x(t)] = T \left[ \int_{-\infty}^{\infty} x(s)\delta(t - s)\,ds \right] \]

\[ = T \left[ \lim_{\Delta \to 0} \sum_{k=\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta \right] \]

\[ = \lim_{\Delta \to 0} \sum_{k=\infty}^{\infty} T \left[ x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta \right] \quad \text{(Additivity)} \]

\[ = \int_{-\infty}^{\infty} T \left[ x(s)\delta(t - s) \right]\,ds \]

\[ = \int_{-\infty}^{\infty} x(s)T[\delta(t - s)]\,ds \quad \text{(Homogeneity)} \]
Convolution and the Impulse Response Function

Let \( h(t) = T[\delta(t)] \) represent the impulse response function. Then

\[
y(t) = \int_{-\infty}^{\infty} x(s)h(t - s)\,ds \triangleq x(t) \ast h(t)
\]

where \( \ast \) signifies convolution.
Properties of Convolution

\[ x * y = y * x \quad \text{(commutative)} \]
\[ (x * y) * z = x * (y * z) \quad \text{(associative)} \]
\[ (x * z) + (y * z) = (x + y) * z \quad \text{(distributive)} \]
A system is linear and shift-invariant if and only if the output is a weighted sum of the input.

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**Output (Impulse Response)**

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### Input (Step)

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**Output (Step Response)**

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Fourier Series

- We have already seen that any signal $x(t)$ can be expressed exactly as an infinite sum of impulses.
- It turns out that any signal can alternatively be expressed exactly as an infinite sum of sinusoids.
- This is known as a Fourier series.

$$x(t) = \int_{0}^{\infty} A_f \sin(2\pi ft + \phi_f) df = \int_{0}^{\infty} A_\omega \sin(\omega t + \phi_\omega) d\omega$$

Fourier Series Approximations

Original Squarewave

4 Term Approximation

8 Term Approximation

16 Term Approximation
Fourier Transforms

- The expansion of a signal in terms of sinusoids can be more neatly expressed using complex numbers:

\[ x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df \]

Inverse Fourier transform \( x(t) = \mathcal{F}^{-1}[X(f)] \)

where

\[ e^{j2\pi ft} = \cos 2\pi ft + j \sin 2\pi ft \] (Euler’s equation)

- \( X(f) \) can be computed from \( x(t) \) using

\[ X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \]

Fourier transform \( X(f) = \mathcal{F}[x(t)] \)
The Fourier transform of sinusoidal signals of two different frequencies.

The complex exponential notation, remember, is just a shorthand for sinusoids and cosinusoids, but it is mathematically more convenient. The Fourier transform coefficients are complex numbers and can be expressed either in terms of their real and imaginary parts, or in terms of their amplitude and phase.

The Fourier transform was one of the tricks that made it easy to write the transforms of both sides of Eq. 6.

The convolution property of the Fourier transform, is that the Fourier transform of the convolution of two functions is the product of their Fourier transforms. This is really just a restatement of the convolution theorem.

Do not be put off by the negative frequencies in the plots. The equations for the Fourier transform either in terms of their real and imaginary parts, or in terms of their amplitude and phase. Often, people prefer to plot only the positive frequency components, as was done in Fig. 9, but it is mathematically more convenient. The frequency components of the input signal. This fact is expressed mathematically by the convolution theorem, which you can check for yourself by making sure that Eq. 9 obeys both homogeneity and additivity. This is important because it makes it easy for us to write the Fourier transforms of lots of things. For example, the Fourier transform of a constant is the delta function.

The Fourier transforms of sinusoidal signals of two different frequencies.

Figure 10: The Fourier transform of a sinusoid is a pair of impulses, one at the positive frequency component and one at the negative frequency component.

The linearity of the Fourier transform is the frequency domain. This fact is expressed mathematically by the convolution theorem, which you can check for yourself by making sure that Eq. 9 obeys both homogeneity and additivity. This is important because it makes it easy for us to write the Fourier transforms of lots of things. For example, the Fourier transform of a constant is the delta function.
Some Properties of Fourier Transforms

- The Fourier transform is a linear system. Thus

\[
\mathcal{F}(\alpha x_1(t) + \beta x_2(t)) = \alpha \mathcal{F}(x_1(t)) + \beta \mathcal{F}(x_2(t))
\]

- Convolution property:

\[
\mathcal{F}(h(t) \ast x(t)) = \mathcal{F}(h(t)) \mathcal{F}(x(t)) = H(f)X(f)
\]

- Differentiation property:

\[
\mathcal{F}\left(\frac{d}{dt} x(t)\right) = 2\pi f \mathcal{F}(x(t))
\]
Why Fourier transforms?

- When we feed an impulse into a shift-invariant linear system, we get out a potentially complicated function (the impulse response).
- In contrast, when we feed a sinusoid into a shift-invariant linear system, we always get out another sinusoid of the same frequency, though generally different amplitude and phase.
- Thus to identify the system, all we need to do is stimulate with different sinusoids of known frequency, amplitude and phase, and record the amplitude and phase of each corresponding output sinusoid.
Linear Systems Identification

**Linear Systems Logic**

**Space/time method**

1. Measure the impulse response
2. Input Stimulus
3. Express as sum of scaled and shifted impulses
4. Calculate the response to each impulse
5. Sum the impulse responses to determine the output

**Frequency method**

1. Measure the sinusoidal responses
2. Express as sum of scaled and shifted sinusoids
3. Calculate the response to each sinusoid
4. Sum the sinusoidal responses to determine the output
Example: retinal ganglion cell model

Figure 7: Illustration of an idealized, retinal ganglion-cell receptive field that acts like a bandpass filter (redrawn from Wandell, 1995). This linear on-center neuron responds best to an intermediate spatial frequency whose bright bars fall on-center and whose dark bars fall over the opposing surround. When the spatial frequency is low, the center and surround oppose one another because both are stimulated by a bright bar, thus diminishing the response. When the spatial frequency is high, bright and dark bars fall within and are averaged by the center (likewise in the surround), again diminishing the response.

Figure 8: Block diagrams of linear filters. A: Linear filter with frequency response. B: Bank of linear filters with different frequency responses. C: Feedback linear system.
Different forms of linear system

A

\[ x(\omega) \xrightarrow{h(\omega)} y(\omega) \]

B

A: Linear filter with frequency response

\[ x(\omega) \xrightarrow{h_1(\omega)} y_1(\omega) \]
\[ x(\omega) \xrightarrow{h_2(\omega)} y_2(\omega) \]
\[ \vdots \]
\[ x(\omega) \xrightarrow{h_N(\omega)} y_N(\omega) \]

C

Feedback linear system

\[ x(\omega) \xrightarrow{+} y(\omega) \xrightarrow{f(\omega)} x(\omega) \]